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Two-dimensional Krall–Sheffer polynomials and integrable systems

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Abstract

Two-dimensional Krall–Sheffer polynomials are analogues of the classical orthogonal polynomial. They are eigenfunctions of second-order linear partial differential operators and moreover satisfy orthogonality conditions. We show that all Krall–Sheffer polynomials are connected with two-dimensional superintegrable systems on spaces with constant curvature.

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Assume that $P_n(x, y)$ are polynomials in two variables x, y. As usual, the degree n is the maximal value $n = \max\{i + j\}$ among all possible monomials $x^i y^j$ in the expansion of the polynomial $P_n(x)$.

Krall and Sheffer considered [5] the problem of finding all polynomials $P_n(x, y)$ with the following properties:

(i) The polynomials $P_n(x, y)$ are eigenfunctions of a second-order admissible differential operator L (to be fully defined later)

$$LP_n(x, y) = \lambda_n P_n(x, y) \tag{1}$$

with polynomial coefficients:

$$L = A(x, y)\partial_{xx} + 2B(x, y)\partial_{xy} + C(x, y)\partial_{yy} + D(x, y)\partial_x + E(x, y)\partial_y \tag{2}$$

where $A(x, y), \dots, E(x, y)$ are polynomials in x and y with *real* coefficients. Note that the eigenvalue λ_n depends only on the degree of the polynomial $P_n(x, y)$.

(ii) There exists a nondegenerate linear functional σ defined on the space of all polynomials in two variables such that the orthogonality property

$$\langle \sigma, P_n(x, y) q(x, y) \rangle = 0 \tag{3}$$

holds with $q(x, y)$ any polynomial of degree $<n$. The functional σ can be defined through its moments $\langle \sigma, x^n y^m \rangle = c_{nm}$, where $n, m = 0, 1, 2, \ldots$. By describing a functional as 'nondegenerate' one means that it has the property that if one has $\psi(x, y)\sigma = 0$ for some polynomial $\psi(x, y)$, then $\psi(x, y) \equiv 0$.

The orthogonality property (3) is closely connected with the symmetrizability of the operator L. Recall that the *Lagrange adjoint* of the operator L in equation (2) is defined as [6]

$$L^+ = \partial_{xx} A(x, y) + 2\partial_{xy} B(x, y) + \partial_{yy} C(x, y) - \partial_x D(x, y) - \partial_y E(x, y). \tag{4}$$

The operator L is *symmetric* if $L^+ = L$. The operator L is *symmetrizable* if there exists a real function $\rho(x, y)$ such that the operator $\rho(x, y)L$ is symmetric. As shown in [2], the properties (i), (ii) (given that the functional σ is nondegenerate) imply the symmetrizability of the operator L.

Later on, Engelis [2] independently considered the same problem from a slightly different point of view and found the same classification scheme. In what follows we will use the Engelis scheme which is more convenient for our purposes.

Before presenting the classification scheme given in [2], we recall some facts concerning admissible differential operators L [3, 4, 7].

The differential operator L in equation (2) is called *admissible* if for any positive integer n there exists $n + 1$ linearly independent polynomial eigenvalue solutions of degree n:

$$L Q_n^{(i)}(x, y) = \lambda_n Q_n^{(i)} \qquad i = 0, 1, \ldots, n \tag{5}$$

and there are no polynomial solutions having degree less than n for the same value λ_n.

It can be easily shown that the operator L is admissible if and only if the coefficients $A(x, y), \ldots, E(x, y)$ are of the form [7]

$$\begin{aligned} A(x, y) &= \alpha x^2 + a_{10}x + a_{01}y + a_{00} \\ B(x, y) &= \alpha xy + b_{10}x + b_{01}y + b_{00} \end{aligned} \tag{6}$$

$$\begin{aligned} C(x, y) &= \alpha y^2 + c_{10}x + c_{01}y + c_{00} \\ D(x, y) &= \beta x + d_0 \qquad E(x, y) = \beta y + e_0 \end{aligned} \tag{7}$$

where $\alpha, \beta, a_{ik}, b_{ik}, c_{ik}, d_0, e_0$ $(i, k = 0, 1)$ are arbitrary real parameters with the only restriction $\alpha p + \beta \neq 0$ for $p = 0, 1, 2, \ldots$. The eigenvalues are then

$$\lambda_n = n(\alpha(n-1) + \beta). \tag{8}$$

Note that for admissible polynomials, eigenvalues are nondegenerate, i.e. $\lambda_n \neq \lambda_m$ for $n \neq m$.

There is an obvious geometrical interpretation of the admissible operators (we follow [8]). First of all, perform a similarity transformation of the operator L with some function $\Phi(x, y)$:

$$\begin{aligned} \tilde{L} = \Phi^{-1}(x, y) L \Phi(x, y) &= A(x, y)\partial_{xx} + 2B(x, y)\partial_{xy} + C(x, y)\partial_{yy} \\ &\quad + \tilde{D}(x, y)\partial_x + \tilde{E}(x, y)\partial_y + U(x, y) \end{aligned} \tag{9}$$

where

$$\tilde{D}(x, y) = D(x, y) + 2\frac{A(x, y)\Phi_x(x, y) + B(x, y)\Phi_y(x, y)}{\Phi(x, y)}$$

$$\tilde{E}(x, y) = E(x, y) + 2\frac{C(x, y)\Phi_y(x, y) + B(x, y)\Phi_x(x, y)}{\Phi(x, y)}$$

and

$$U(x, y) = \frac{A\Phi_{xx} + C\Phi_{yy} + 2B\Phi_{xy} + D\Phi_x + E\Phi_y}{\Phi(x, y)}.$$

The operator $\tilde{L}$ (9) can be presented in a form close to that of the Laplace–Beltrami operator associated with a metric $g_{ik}(x, y)$. Indeed, assume that a two-dimensional metric tensor $g_{ik}(x, y)$ is given. This means that for the length element $\mathrm{d}s$ we have $\mathrm{d}s^2 = g_{11}(x, y)\,\mathrm{d}x^2 + g_{22}(x, y)\,\mathrm{d}y^2 + 2g_{12}(x, y)\,\mathrm{d}x\,\mathrm{d}y$. Then, the Laplace–Beltrami operator Δ_{LB} is defined as [1]

$$\Delta_{\mathrm{LB}} = f(x, y)^{1/2}\partial_i g^{ik}(x, y) f(x, y)^{-1/2}\partial_k \tag{10}$$

where $\partial_1 = \partial_x$, $\partial_2 = \partial_y$ and $f(x, y) = \det \|g^{ik}(x, y)\| = \det \|g_{ik}(x, y)\|^{-1}$. From (10) we have

$$\Delta_{\mathrm{LB}} = g^{11}\partial_{xx} + g^{22}\partial_{yy} + 2g^{12}\partial_{xy} + S_1(x, y)\partial_x + S_2(x, y)\partial_y \tag{11}$$

where

$$S_1(x, y) = \frac{\partial g^{11}}{\partial x} + \frac{\partial g^{21}}{\partial y} - \frac{1}{2} f^{-1}(x, y)\left(g^{11}\frac{\partial f(x, y)}{\partial x} + g^{12}\frac{\partial f(x, y)}{\partial y}\right)$$

and

$$S_2(x, y) = \frac{\partial g^{12}}{\partial x} + \frac{\partial g^{22}}{\partial y} - \frac{1}{2} f^{-1}(x, y)\left(g^{12}\frac{\partial f(x, y)}{\partial x} + g^{22}\frac{\partial f(x, y)}{\partial y}\right).$$

Let us compare expression (9) for the operator $\tilde{L}$ with expression (11). It is natural to make the following identifications:

$$g^{11} = A(x, y) \qquad g^{12} = B(x, y) \qquad g^{22} = C(x, y). \tag{12}$$

Then $f(x, y) = A(x, y)C(x, y) - B^2(x, y)$ and we have

$$\tilde{L} = \Delta_{\mathrm{LB}} + T_1(x, y)\partial_x + T_2(x, y)\partial_y + U(x, y) \tag{13}$$

where

$$T_1(x, y) = \tilde{D}(x, y) - A_x(x, y) - B_y(x, y) + \frac{Bf_y(x, y) + Af_x(x, y)}{2f(x, y)}$$

and

$$T_2(x, y) = \tilde{E}(x, y) - B_x(x, y) - C_y(x, y) + \frac{Bf_x(x, y) + Cf_y(x, y)}{2f(x, y)}.$$

So, under the identification (12) we see from (13) that the operator $\tilde{L}$ coincides with the Laplace–Beltrami operator Δ_{LB} up to terms containing only the first derivatives and the 'potential' $U(x, y)$.

It is natural to ask whether a function $\Phi(x, y)$ exists such that the condition

$$T_1(x, y) = T_2(x, y) \equiv 0 \tag{14}$$

holds.

If (14) is valid, then we have

$$\tilde{L} = \Delta_{\mathrm{LB}} + U(x, y). \tag{15}$$

On the other hand, it is well known [1] that the Laplace–Beltrami operator Δ_{LB} plays the role of the free-motion Hamiltonian for a quantum mechanical particle on a Riemannian space with the metric $g_{ik}(x, y)$. Hence, condition (14) says that the operator $\tilde{L}$ coincides with the Schrödinger operator on this Riemannian space with the potential $U(x, y)$.

If condition (14) cannot be satisfied then it can be interpreted as indicating the presence of a *magnetic field*. Therefore, condition (14) reflects the absence of magnetic fields.

It is easily seen [8] that condition (14) is equivalent to the condition

$$\partial_y\left(\frac{CK_1 - BK_2}{f}\right) = \partial_x\left(\frac{AK_2 - BK_1}{f}\right) \tag{16}$$

where

$$K_1(x, y) = \frac{4A_x f + 4B_y f - Af_x - Bf_y - D}{4f}$$
$$K_2(x, y) = \frac{4B_x f + 4C_y f - Bf_x - Cf_y - E}{4f}.$$

We see that any admissible operator L depends on 13 parameters. It is natural to call the ten parameters $\alpha, a_{ik}, b_{ik}, c_{ik}$ *internal parameters*. Indeed, these parameters define the metric tensor $g_{ik}(x, y)$ and hence describe the geometrical properties of the operator L. The remaining three parameters β, d_0, e_0 will be called *external parameters*. These parameters describe the interaction of our system with external fields.

Of course, it is possible to reduce the number of independent internal parameters by means of affine transformations of the independent arguments x, y. We will describe this procedure following [7].

Consider all invertible *affine* transformations of the form

$$\begin{aligned} x &= q_{11}\xi + q_{12}\eta + q_{10} \\ y &= q_{21}\xi + q_{22}\eta + q_{20} \end{aligned} \tag{17}$$

with some coefficients q_{ik}. Then it is easily shown [7] that if L is admissible in coordinates x, y, then L remains admissible in the new coordinates ξ, η. Moreover, property (ii) and the symmetrizability of the operator L are also preserved under the transformation (17). Property (14) is also preserved under affine transformations. This means that if the operator L is reduced to the form (15) without magnetic field then the affine-transformed operator L can also be reduced to the same form.

The parameter α is preserved under affine transformations. Hence we can make a division into two cases: $\alpha \neq 0$ and $\alpha = 0$. If $\alpha \neq 0$ we can put $\alpha = 1$ without loss of generality. Indeed, in this case we can divide the lhs and rhs of equation (1) by α. This will lead only to a renormalization of the remaining nine internal and three external parameters.

As affine transformation (17) contains six independent parameters, it is possible to reduce the nine internal parameters a_{ik}, b_{ik}, c_{ik} to three independent parameters. We have thus achieved a division of the admissible operators L into two classes: those with $\alpha \neq 0$ and those with $\alpha = 0$; and each class contains six independent parameters: three internal and three external ones.

The characteristic determinant $f(x, y) = \det \|g^{ik}(x, y)\| = A(x, y)C(x, y) - B^2(x, y)$ plays a crucial role in the classification of all possible distinct cases of admissible operators (for details see, e.g., [7]).

We can formulate the main result of [2] as follows.

Theorem 1. *If the operator L is admissible and there exists a nondegenerate functional σ, then the operator L is symmetrizable. Moreover, up to affine transformation, there exist nine distinct types of L:*

(I) $A(x, y) = x^2 - x$, $B(x, y) = xy$, $C(x, y) = y^2 - y$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$; *the characteristic determinant is* $f(x, y) = xy(1 - x - y)$.

(II) $A(x, y) = x^2$, $B(x, y) = xy$, $C(x, y) = y^2 - y$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$, $f(x, y) = -x^2 y$.

(III) $A(x, y) = x^2$, $B(x, y) = xy$, $C(x, y) = y^2 + x$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$, $f(x, y) = x^3$.

(IV) $A(x, y) = -x$, $B(x, y) = 0$, $C(x, y) = -y$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$, $f(x, y) = xy$.

(V) $A(x, y) = 0$, $B(x, y) = x$, $C(x, y) = y$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$, $f(x, y) = -x^2$.

(VI) $A(x, y) = -x$, $B(x, y) = 0$, $C(x, y) = -1$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$, $f(x, y) = x$.

(VII) $A(x, y) = -1$, $B(x, y) = 0$, $C(x, y) = -1$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$, $f(x, y) = 1$.

(VIII) $A(x, y) = y$, $B(x, y) = 1$, $C(x, y) = 0$, $D(x, y) = \beta x + d_0$, $E(x, y) = \beta y + e_0$, $f(x, y) = -1$.

(IX) $A(x, y) = x^2 - 1$, $B(x, y) = xy$, $C(x, y) = y^2 - 1$, $D(x, y) = \beta x$, $E(x, y) = \beta y$, $f(x, y) = 1 - x^2 - y^2$.

As was shown in [8], in all nine cases the Krall–Sheffer–Engelis (KSE) operators L can be transformed into a form in which they describe integrable quantum mechanical systems on spaces of constant curvature without magnetic field.

Direct computation yields [8]:

Proposition 1. *Condition (16) holds for every case (I)–(IX) of the KSE classification scheme. Hence every case can be transformed to a quantum system on two-dimensional manifolds with some potential $U(x, y)$ without magnetic field.*

Our next step will be to find the mean Riemannian curvature $\kappa(x, y)$ corresponding to the metric $g_{ik}(x, y)$. The Riemannian curvature $\kappa(x, y)$ can be calculated from the components $g_{ik}(x, y)$ of the metric using standard formulae from differential geometry. Performing these simple calculations we arrive at the following:

Proposition 2. *The mean Riemannian curvature is constant for every case (I)–(IX) of the KSE classification scheme. More precisely, the cases (IV)–(VIII) correspond to zero curvature, whereas the cases (I)–(III) and (IX) correspond to a nonzero curvature.*

For details and examples of corresponding quantum systems see [8].

Thus all nine types correspond to some quantum mechanical systems describing the motion of a particle in the presence of some potentials on two-dimensional spaces of constant curvature. There are no magnetic fields in any of these cases.

We now present the main result of this paper. That is, we show that all nine types in the KSE classification correspond to superintegrable systems. This means that there exist two algebraically independent operators I_1 and I_2 commuting with the operator L: $[L, I_1] = [L, I_2] = 0$. Operators $I_{1,2}$ act on the space of polynomials of two variables and preserve the degree of a polynomial.

Theorem 2. *For all nine types in the KSE classification scheme the algebraically independent integrals I_1, I_2 commuting with the operator L are*

Case (I).

$$I_1 = x(1 - x - y)\partial_{xx} + (d_0(y - 1) - (\beta + e_0)x)\partial_x$$
$$I_2 = y(1 - x - y)\partial_{yy} + (e_0(x - 1) - (\beta + d_0)y)\partial_y.$$

Case (II).

$$I_1 = x^2\partial_{xx} + ((\beta + e_0)x + d_0(1 - y))\partial_x$$
$$I_2 = xy\partial_{yy} + (d_0 y - e_0 x)\partial_y.$$

Case (III).

$$I_1 = x^2\partial_{yy} + (e_0 x - d_0 y)\partial_y$$
$$I_2 = 2x^2\partial_{xy} + xy\partial_{yy} + (e_0 x - d_0 y)\partial_x + (\beta x + d_0)\partial_y.$$

Case (IV).

$$I_1 = -x\partial_{xx} + (\beta x + d_0)\partial_x$$
$$I_2 = -xy(\partial_x - \partial_y)^2 + (d_0 y - e_0 x)\partial_x + (e_0 x - d_0 y)\partial_y.$$

Case (V).

$$I_1 = x^2\partial_{xx} + (e_0 x - d_0 y)\partial_x$$
$$I_2 = x\partial_{yy} + (\beta x + d_0)\partial_y.$$

Case (VI).

$$I_1 = -x\partial_{xx} + (\beta x + d_0)\partial_x$$
$$I_2 = \partial_{yy} - (e_0 + \beta y)\partial_y.$$

Case (VII).

$$I_1 = -\partial_{xx} + (\beta x + d_0)\partial_x$$
$$I_2 = (d_0 + \beta x)\partial_y - (\beta y + e_0)\partial_x.$$

Case (VIII).

$$I_1 = \partial_{xx} + (\beta y + e_0)\partial_x$$
$$I_2 = (x - y^2)\partial_{xx} - 2y\partial_{xy} - \partial_{yy} + (e_0 x - d_0 y)\partial_x - (\beta x + d_0)\partial_y.$$

Case (IX).

$$I_1 = x\partial_y - y\partial_x$$
$$I_2 = (1 - x^2 - y^2)\partial_{xy} + (1 - \beta)x\partial_y.$$

Note that in cases (VII) and (IX) there exist integrals of first order. In all other cases we have integrals of second order with respect to the derivatives.

Thus all types (I)–(IX) correspond to superintegrable systems. Recall that a two-dimensional system is called integrable if there exists at least one integral commuting with the Hamiltonian. Superintegrable systems (with two algebraically independent integrals) form a subclass of integrable systems.

Our next main result is a characterization of the existence of a nondegenerate orthogonality functional in terms of integrability.

Theorem 3. *The existence of a nondegenerate orthogonality functional for an admissible operator L is equivalent to the existence of a second-order integral I commuting with L: $[L, I] = 0$.*

The proof of this statement is based on a direct computation of second-order integrals for admissible operators. It appears that the existence of such integrals imposes additional restrictions on the values of intrinsic parameters. We saw that affine transformations allow one to reduce the number of intrinsic parameters to three. Direct calculations show that the existence of commuting integrals reduces this number to zero. That is, there are three additional conditions fixing these three parameters. This leads (up to affine transformations) to the same coefficients $A(x, y)$, $B(x, y)$, $C(x, y)$ as in the KSE classification scheme. Details of the proof will be published separately.

The meaning of this theorem is that all nine cases in the KSE scheme can be equally characterized by the existence of at least one second-order integral. Note that superintegrability (i.e. the existence of the second independent integral) is obtained as a by-product during the proof of this theorem.

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References

[1] Barut A O and Rączka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN)

[2] Engelis G K 1974 On two-dimensional analogues of classical orthogonal polynomials *Latviiskii Matem. Ezhegodnik, Vyp.* **15** 169–202 (in Russian)

[3] Kim Y J, Kwon K H and Lee J K 1997 Orthogonal polynomials in two variables and second-order partial differential equations *J. Comput. Appl. Math.* **82** 239–60

[4] Kim Y J, Kwon K H and Lee J K 1998 Partial differential equations having orthogonal polynomial solutions *J. Comput. Appl. Math.* **99** 239–53

[5] Krall H and Sheffer I M 1967 Orthogonal polynomials in two variables *Ann. Math. Pure Appl.* **76** 325–76

[6] Littlejohn L L 1988 Orthogonal polynomial solutions to ordinary and partial differential equations *Proc. 2nd Int. Symp. on Orthogonal Polynomials and their Applications (Segovia, Spain, 1986) (Springer Lecture Notes in Mathematics vol 1329)* ed M Alfaro *et al* (Berlin: Springer) pp 98–124

[7] Suetin P K 1988 *Orthogonal Polynomials in Two Variables* (Moscow: Nauka) (in Russian)

[8] Vinet L and Zhedanov A 2000 Two-dimensional Krall–Sheffer polynomials and quantum systems on spaces with constant curvature *CRM Preprint*